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Ground states for a class of deterministic spin models with glassy behaviour

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Abstract. We consider the deterministic model with glassy behaviour, recently introduced by Marinari, Parisi and Ritort, with Hamiltonian $H = \sum_{i,j=1}^{N} J_{i,j}\sigma_i\sigma_j$, where J is the discrete sine Fourier transform. The ground state found by these authors for N odd and $2N+1$ prime is shown to become asymptotically degenerate when $2N+1$ is a product of odd primes, and to disappear for N even. This last result is based on the explicit construction of a set of eigenvectors for J, obtained through its formal identity with the imaginary part of the propagator of the quantized unit symplectic matrix over the 2-torus.

1. Introduction

It has been recently established [1–3] that a wide class of deterministic, infinite-range deterministic Ising spin models does actually exhibit the glassy behaviour of the random coupling case, with the important difference, however, that the mean-field equations of the model, derived by Parisi and Potters [4] (hereafter called the PP equations), are not the standard TAP equations. Unlike the random case (see, e.g., [5]), they do not determine the critical temperature of the glassy transition by linearization around the largest eigenvalue of the interaction matrix.

Among these models a special role is played by the so-called *sine* (or, equivalently, *cosine*) model, in which the interaction matrix $J = (J)_{i,j}, i, j = 1, \ldots, N; N \in \mathbb{N}$ among N spins (with periodic boundary conditions) defining the Hamiltonian

$$H = -\frac{1}{2}\sum_{i,j=1}^{N} J_{i,j}\sigma_i\sigma_j \tag{1.1}$$

is given by

$$J_{i,j} = \frac{2}{\sqrt{2N+1}}\sin\left(\frac{2\pi ij}{2N+1}\right) \qquad i, j = 1, \ldots, N \tag{1.2}$$

namely by twice the uppermost left block of the the discrete sine (cosine) Fourier transform

$$S_{i,j} = \frac{1}{\sqrt{2N+1}}\sin\left(\frac{2\pi ij}{2N+1}\right) \qquad i, j = 1, \ldots, 2N\,. \tag{1.3}$$

The factor 2 accounts for the orthogonality of J. Here N is odd and $p = 2N + 1$ prime, i.e. p is a prime of the form $p = 4m+3$. In fact, in this case the ground state configuration

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can be explicitly computed [3] and is given by $\sigma_j = \left(\frac{j}{p}\right)$ $j = 1, \ldots, N$. Here $\left(\frac{j}{p}\right)$ is the Legendre symbol of j, namely

$$\left(\frac{j}{p}\right) = \begin{cases} +1 & \text{if } j \equiv x^2 \pmod p \\ -1 & \text{if } j \not\equiv x^2 \pmod p \end{cases} \tag{1.4}$$

where $x \in (0, 1, \ldots, p-1) \equiv \mathbb{Z}_p = \mathbb{Z}(\bmod\ p)$. In other words, if j is a quadratic residue of p its Legendre symbol is 1, and -1 in the opposite case. The existence of such a complex ground state, proved by showing that on the spin configuration defined by the Legendre symbols the energy actually assumes its absolute minimum $-\frac{1}{2}N$, yields the possibility of numerically detecting a first-order 'crystalline' phase transition at a temperature higher than the critical one for the glassy transition [3, 4] and of explicitly finding [4] the corresponding solution of the PP mean-field equations under the form $m_i = \sqrt{q}\left(\frac{i}{p}\right)$. Here as usual m_i denotes the magnetization on the site i, and $q = \frac{1}{N}\sum_{i=1}^N m_i^2$ the Edwards–Anderson order parameter. The reader is referred to [4, section 3] for a discussion of the relevance of the existence of crystalline phase on the glassy behaviour of the system.

The existence of such a ground state critically depends on the arithmetic restrictions on N (actually Parisi and Potters [4] give arguments supporting its disappearance for general values of N) and hence it can be of interest to look into the question from a rigorous point of view.

We can distinguish two cases for p odd:

A. p is of the form $4m+3$ (the case considered by Marinari *et al* when p is prime);
B. p is of the form $4m+1$.

In case A we show that, when the restriction on the primality of p is essentially removed, namely when $p = 2N+1 = p_1 p_2 \cdots p_s$ is the product of s distinct primes such that $2N+1 = 4m+3$ (the factorization of $p = 2N+1$ consists of an odd number t of primes p_i of the form $4m_i+3$ and of an arbitrary number of primes of the form $p_i = 4m_i+1$), then

(i) The ground-state energy $-\frac{1}{2}N$ becomes asymptotically degenerate of order $D = \mathrm{O}(2^{N^{(s-1)/s}})$, namely there are D distinct spin configurations $\sigma_j^{(l)} : j = 1, \ldots, p,\ l = 1, \ldots, D$ such that their energy $E(\sigma_j^{(l)})$ fulfills the estimate

$$E(\sigma_j^{(l)}) \leqslant -\tfrac{1}{2}N\left(1 - KN^{-1/s}\right) \tag{1.5}$$

for some positive constant K independent of l.

(ii) The D distinct spin configurations $\sigma_j^{(l)}$ are obtained as follows:

$$\sigma_j^{(l)} = \begin{cases} \psi(j) & \text{if } \psi(j) \neq 0 \\ \pm 1 & \text{if } \psi(j) = 0 \end{cases} \tag{1.6}$$

where $\psi(j)$ is the Jacobi symbol of $j \in \mathbb{Z}_p$ with respect to $p = 2N+1$:

$$\psi(j) = \prod_{i=1}^{s}\left(\frac{j}{p_i}\right) \qquad \left(\frac{j}{p_i}\right) = 0 \quad \text{if } (j, p_i) = 0 \tag{1.7}$$

(here (k, p) denotes the MCD between k and p; $(k, p) = 0$ means that k is a multiple of p) and the number of zeroes of $\psi(j)$ behaves like $N^{(s-1)/s}$ for large N.

(iii) For q suitably small, the magnetization vectors $m_j^{(l)} = \sqrt{q}\sigma_j^{(l)}$ solve the PP mean-field equations

$$m_i = \tanh\left(2\beta G'(\beta(1-q))m_i - \beta\sum_{j=1}^{p} J_{ij}m_j\right) \tag{1.8}$$

where

$$G(x) = -\frac{1}{4}\ln\left(\frac{1+\sqrt{1+4x^2}}{2}\right) + \frac{1}{4}\left(\sqrt{1+4x^2}-1\right) .$$

We will recall later how the above properties allow us an immediate application of the argument of Parisi and Potters [4, section 4] to strongly support the conclusion that in this case there are D 'crystal' states with properties analogous to the one existing for $p = 4m+3$ prime.

In case B we consider the case analogous to that of [3], namely $p = 2N + 1$ prime of the form $4m + 1$, and show that J cannot admit eigenvectors whose components are all of the form ± 1 in correspondence to the eigenvalues ± 1. Hence the minimum of the energy quadratic form $\langle u, Ju\rangle$ is never reached when u is a spin configuration, let alone the configuration of the Legendre symbol valid for $p = 4m + 3$. This represents (for such values of N) a rigorous proof of the conjecture of [4], and hence suggests disappearance of the above 'crystal state' picture. As remarked also by G Parisi (private communication) this result represents a strong indication that there could be two different thermodynamic limits of the phase diagram according to the arithmetic properties of the number of spins N in finite volume.

In section 2 we describe the number theoretic argument proving properties (i), (ii) and (iii) of case A, and in section 3 we prove case B, basing the proof on the explicit construction of a set of eigenvectors for the operator J.

This construction can be interesting in itself because it is based on the metaplectic representation of the quantized symplectic linear maps over the 2-torus. In particular, the operator S turns out to coincide [10, 11] with (the imaginary part of) the operator quantizing of the standard unit symplectic 2×2 matrix

$$I_{\mathrm{sp}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This operator is a $N \times N$ unitary matrix, N being the inverse of the Planck constant, so that in this context the thermodynamic limit $N \to \infty$ is formally equivalent to the classical limit.

2. A real eigenvector of the operator J

Marinari *et al* [3] prove that for $p = 4m + 3 = 2N + 1$ the ground state is given by the spin configuration defined by the Legendre symbols

$$\sigma_L = (\sigma_1, \ldots, \sigma_N) \qquad \sigma_j = \left(\frac{j}{p}\right) \quad j = 1, \ldots, N$$

by explicit verification that σ_L is an absolute minimizer of the energy. As a consequence, σ_L must necessarily be an eigenvector of the operator J defined by (1.2) corresponding to the eigenvalue 1 (recall that J is a real orthogonal matrix so that its spectrum consists only of the eigenvalues ± 1). In analogy with this result, in the present case the basic step is represented by the construction of an eigenvector of the operator J whose components are all ± 1 or 0 because it is defined by the Jacobi symbol $\chi_J(x) = \prod_{i=1}^{s}\left(\frac{x}{p_i}\right)$, $x = 1, \ldots, p$.

The construction of this eigenvector will be an easy consequence of the following lemma.

Lemma 2.1. Let $N = \prod_{i=1}^{s} p_i$ be the product of s pairwise different odd primes such that $N = 4m + 3$. Then the matrix S defined by (1.3), whose elements are

$$S_{k,x} = \frac{1}{\sqrt{N}} \sin\left(\frac{2\pi}{N} kx\right) \tag{2.1}$$

admits the vector $\chi_J(x) = \prod_{i=1}^{s} \left(\frac{x}{p_i}\right)$, $x = 1, \ldots, N$ as an eigenvector corresponding to the eigenvalue 1.

Proof. It is a classical result in number theory (see, e.g., [7, proposition A.7]) that $\chi_J(x), x \in \mathbb{Z}_N$ is the unique primitive multiplicative character of the ring $\mathbb{Z}_N = \mathbb{Z}(\mathrm{mod} N)$, and it is also well known (see again [7, proposition 2.1] or [8, theorem 8.15]) that the Gaussian sum

$$\tau_k(x) = \sum_{k=1}^{N} \chi(x) \mathrm{e}^{(2\pi i/N)kx} \tag{2.2}$$

is separable for all k if χ is a primitive multiplicative character. Namely, one has

$$\tau_k(\chi) = \sum_{k=1}^{N} \overline{\chi}(k) \tau_1(\chi) \tag{2.3}$$

where $\overline{\chi}(k)$ denotes the complex conjugate of $\chi(k)$. On the other hand, [7, theorem 2.1] states that, if χ is any real primitive character

$$\tau_1(\chi) = \begin{cases} N^{\frac{1}{2}} & \text{if } \chi(-1) = 1 \\ i N^{\frac{1}{2}} & \text{if } \chi(-1) = -1 \,. \end{cases} \tag{2.4}$$

Therefore in our case we get

$$(S\chi_J)_k = \frac{1}{\sqrt{N}} \sum_{x=1}^{N} \sin\left(\frac{2\pi}{N} kx\right) \chi_J(x) = \frac{1}{2i\sqrt{N}} (\chi_J(k) - \chi_J(-k)) \sum_{x=1}^{N} \chi_J(x) \mathrm{e}^{(2\pi i/N)kx}$$

$$= \frac{1}{2i\sqrt{N}} (\chi_J(k) - \chi_J(-k))\, i\sqrt{N} = \frac{1}{2} (\chi_J(k) - \chi_J(-k))$$

since $\chi_J(-1) = -1$ if $N = 4m + 3$ (see, e.g., [8, theorem 9.10]). Now, if $(k, p_i) > 1$ for at least one i, then $\chi_J(-k) = 0 = \chi_J(k)$ by definition. Let now $(k, p_i) = 1$ for all $1 \leqslant i \leqslant s$. By the multiplicative property of the Legendre symbols we have

$$\left(\frac{-k}{p_i}\right) = \left(\frac{-1}{p_i}\right) \cdot \left(\frac{k}{p_i}\right) = -\left(\frac{k}{p_i}\right) \qquad \left(\frac{-k}{p_j}\right) = \left(\frac{-1}{p_j}\right) \cdot \left(\frac{k}{p_i}\right) = \left(\frac{k}{p_i}\right)$$

because -1 is quadratic residue of all primes of the form $4m + 1$ and non residue of the primes of the form $4m + 3$ [8, theorem 9.10]. Hence we can write

$$\chi_J(-k) = \prod_{\substack{i=1 \\ p_i = 4m_i + 3}}^{t} \left(\frac{-k}{p_i}\right) \cdot \prod_{\substack{j=1 \\ p_j = 4m_j + 1}}^{s-t} \left(\frac{-k}{p_j}\right)$$

$$= (-1)^t \prod_{\substack{i=1 \\ p_i = 4m_i + 3}}^{t} \left(\frac{k}{p_i}\right) \cdot \prod_{\substack{j=1 \\ p_j = 4m_j + 1}}^{s-t} \left(\frac{k}{p_j}\right) = -\chi_J(k) \,.$$

Therefore

$$(S\chi_J)(k) = \tfrac{1}{2}(\chi_J(k) - \chi_J(-k)) = \chi_J(k)$$

and this concludes the proof of the lemma. □

We proceed now to the construction of the real eigenvector of J with components $(\pm 1, 0)$.

Corollary 2.1. Let N be such that $p \equiv 2N + 1 = \prod_{i=1}^{s} p_i$ where $p_i : i = 1, \ldots, s$ are s distinct primes such that there is an odd number t of primes p_i of the form $4m_i + 3$. Let $\psi \in \mathbb{R}^n$ be the vector formed by the first N components of the real primitive character $\chi_J(x)(\bmod\ p = 2N + 1)$ defined in lemma 2.1 above. Then the operator J acting on C^N, defined by the matrix (1.2), admits ψ as an eigenvector corresponding to the eigenvalue 1.

Proof. We have

$$J_{k,x} = \frac{2}{\sqrt{2N+1}} \sin\left(\frac{2\pi kx}{2N+1}\right).$$

Now the matrix S is row antisymmetric, and the $(2N + 1)$th row vanishes. We have seen above that $\chi_J(-k) = -\chi_J(k)$. Then, by the former lemma

$$\begin{aligned}(J\psi)_k &= \frac{2}{\sqrt{2N+1}} \sum_{x=1}^{N} \sin\left(\frac{2\pi kx}{2N+1}\right) \psi(k)\\ &= \frac{2}{2\sqrt{2N+1}} \sum_{x=1}^{2N+1} \sin\left(\frac{2\pi kx}{2N+1}\right) \psi(k) = \chi_J(k)\end{aligned}$$

where we now have $k = 1, \ldots, N$. This proves the corollary. □

Let us now turn to the proof of assertions (i), (ii) and (iii) stated in the introduction. Consider first the simplest possible case, given by $N = p_1 p_2$, where p_1 is of the form $4m + 3$ and p_2 of the form $4m + 1$, so that N is of the form $4m + 3$. We can assume (see, e.g., [7]) that $|p_1 - p_2|$ is independent of N, so that $p_1 \sim \sqrt(N)$, $p_2 \sim \sqrt(N)$ as $N \to \infty$. With $p = 2N + 1$, consider the eigenvector $\psi(x) : \psi(x) \in \{-1, 0, 1\}$ of corollary 2.1. Remark that the zero components of the eigenvector are obtained in correspondence of the multiples of p_1 or p_2 between 1 and N. There are at most

$$h = \frac{p_1 - 1}{2} + \frac{p_2 - 1}{2}$$

such multiples, and, since $p_{1,2} \sim \sqrt{N}$, we have $h \sim \sqrt{N}$. Hence the energy $E(\psi)$ of the vector ψ (note that this vector is not a spin configuration) is given by

$$\begin{aligned}E(\psi) &= -\frac{1}{2} \sum_{i,j=1}^{N} S_{i,j} \psi_i \psi_j = -\frac{1}{2}\left(|\psi^+|^2 - |\psi^-|^2\right)\\ &\sim -\frac{1}{2}(N - \sqrt{N}) = -\frac{N}{2}\left(1 - \frac{1}{\sqrt{N}}\right).\end{aligned}$$

Here ψ^+ and ψ^- denote the projection of ψ on the eigenspaces $V^\pm$ corresponding to the eigenvalues 1 and -1, respectively.

Now out of ψ we can define $D = 2^h$ spin configurations in the following way:

$$\sigma_x = \begin{cases} \psi(x) & \text{if } \psi(x) \neq 0 \\ \pm 1 & \text{if } \psi(x) = 0 \end{cases} \qquad x = 1, \ldots, N\,. \tag{2.5}$$

Now set $v = \sigma - \psi$. The energy $E(\sigma) = -\frac{1}{2}\left(\|\sigma^+\|^2 - \|\sigma^-\|^2\right)$ is obviously maximal when $v \in V^-$. Therefore, since v has at most $\sqrt{N}$ non-zero components and ψ is an eigenvector corresponding to the eigenvalue 1 of S, we have

$$E(\sigma) \leqslant -\frac{1}{2}\left(\|\psi\|^2 - \|v\|^2\right) \sim -\frac{1}{2}(N - \sqrt{N}) = -\frac{N}{2}\left(1 - \frac{1}{\sqrt{N}}\right)\,.$$

Hence the energy of all D states $\psi^l : l = 1, \ldots, D$ tend to the minimum energy $-\frac{1}{2}N$ as $N \to \infty$.

There is now no difficulty in extending the argument to the general case stated in section 1, in which $N = p_1 \times \cdots \times p_s$ with $N = 4m + 3$ and $p_1 < p_2 < \cdots < p_s$ odd primes. We assume that there is a constant C (depending on s) such that $p_s \leqslant C p_1$. By repeating the above argument one easily obtains that in this case the number h of the zero components of the eigenvector ψ of S fulfills the estimate

$$h \sim A N^{(s-1)/s} \tag{2.6}$$

for some constant A indepedent of N (but dependent on s). Hence, as above, we can construct $D = 2^h$ spin configurations σ whose energy $E(\sigma)$ fulfills the estimate

$$E(\sigma) \leqslant -\frac{N}{2}\left(1 - \frac{K}{N^{1/s}}\right) \tag{2.7}$$

for some K independent of s, and thus the ground state is asymptotically degenerate of order D as $N \to \infty$. This concludes the verification of assertions (i) and (ii) of section 1.

The verification of assertion (iii) proceeds exactly as in [4]: the ansatz

$$m_i = \sqrt{q}\epsilon_i \qquad q = \frac{1}{N}\sum_{i=1}^{N} m_i^2$$

where the $\{\epsilon_i\}$ are ± 1 or 0 reduces the PP equations (1.8) to

$$q = \tanh^2\left\{\beta\sqrt{q}\left[1 + \frac{1 - \sqrt{1 + 4\beta^2(1-q)^2}}{2\beta(1-q)}\right]\right\}\,. \tag{2.8}$$

Since we can take for $\{\epsilon_i\}$ any one of the eigenvectors ψ^l; $l = 1, \ldots, D$ of S constructed before, we see that the magnetization vectors of components $m_i^l = \sqrt{q}\psi_i^l$ yield D solutions of the mean-field equations (1.8) provided q solves (2.8). Now equation (2.8) always admits the paramagnetic solution $q = 0$ and hence, as in [4], for sufficiently large β will also admit a solution for $q \neq 0$. Moreover, the specific Gibbs free energy βf_l of all solutions will be given by [4, equation (26)] up to an error of order $N^{-1/s}$, namely

$$\beta f_l = \frac{1+\sqrt{q}}{2}\ln\left[\frac{1}{2}(1+\sqrt{q})\right] + \frac{1-\sqrt{q}}{2}\ln\left[\frac{1}{2}(1-\sqrt{q})\right] - \frac{\beta}{2}q - G(\beta(1-q)) + \mathrm{O}(N^{-1/s})\,. \tag{2.9}$$

In fact, the total Gibbs free energy $\beta\Phi$ as a function of the magnetizations m_i is given by [4, equation (19)]

$$\beta\Phi = \frac{1}{2}\sum_{i=1}^{N}\left\{(1+m_i)\ln\left[\frac{1}{2}(1+m_i)\right] + (1-m_i)\ln\left[\frac{1}{2}(1-m_i)\right]\right\}$$
$$-\frac{\beta}{2}\sum_{i,j=1}^{N} S_{i,j}m_im_j - NG(\beta(1-q)) .$$

Taking $m_i = \sqrt{q}\psi_i^l$ we get (2.9) because $\sum_{i,j=1}^{N} S_{i,j}\psi_i^l\psi_j^l = q + \mathrm{O}(N^{-1/s})$.

Therefore we can apply directly the results of the numerical analysis of [4] showing that (2.8) admits a solution with $q = 0.92$ for $T < 0.400$ to conclude that there are D solutions with such q, which for sufficiently large N will have free energy larger than that of the paramgnetic solutions as long as $T > 0.178$, and smaller for $T < 0.178$ so that the absolute minimum of the specific free energy is also asymptotically degenerate. Therefore we can conclude that the picture of the 'crystal' state should persist also in this situation, up to a degeneracy.

3. The case of p prime of the form $4m + 1$. Explicit construction of the eigenvectors of S

Let us first proceed to the construction of a set of eigenvectors for S. We start from the obvious observation that this operator is the imaginary part of the discrete Fourier transform, defined as

$$\mathcal{F}\psi_l = \frac{1}{\sqrt{p}}\sum_{k=0}^{p-1} \mathrm{e}^{2\pi i/p}\psi_k . \tag{3.1}$$

Namely

$$S = \frac{\mathcal{F} - \mathcal{F}^{-1}}{2i} = \frac{\mathcal{F} - \mathcal{F}^{*}}{2i} . \tag{3.2}$$

The discrete Fourier transform operator $\mathcal{F}$ coincides with the unitary evolution operator V_J quantizing, via canonical (see [10]) or, equivalently, geometric (see [11]) quantization and metaplectic representation of $\mathrm{Sp}(1, \mathbb{R})$, the map on the torus $\mathbb{T}^2$ defined by the standard unit symplectic matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.3}$$

This enables us to adapt to the elliptic map of the present case the eigenvector construction obtained in [12] for the hyperbolic ones, based on the determination of suitable linear combinations of the orthogonal vectors (for fixed $k \in \mathbb{Z}_N$)

$$\psi_{k,l}(q) = \frac{1}{\sqrt{p}}\exp\left[\frac{2\pi i}{p}(kq^2 + lq)\right] \qquad k, l \in \mathbb{Z}_p \tag{3.4}$$

by action of the map itself.

The orthonormality of the basis $\{\psi_{k,l}(q)\} : l = 0, \ldots, p-1$, k fixed requires p prime and can be easily deduced using the well known properties of quadratic Gauss sums, in particular from the relation (see e.g., [8, chapter 9])

$$\sum_{k=0}^{p-1} \exp\left[\frac{2\pi i}{p}(ak^2+bk)\right] = \begin{cases} \epsilon_p p^{\frac{1}{2}} \left(\frac{a}{p}\right) \exp\left[\frac{2\pi}{p} b^2 (4a)^{-1}\right] & \text{if } a \not\equiv 0 \pmod{p} \\ p\delta_b^0 & \text{if } a \equiv 0 \pmod{p} \end{cases} \tag{3.5}$$

where

$$\epsilon_p = \begin{cases} 1 & p \equiv 1 \pmod 4 \\ i & p \equiv 3 \pmod 4 . \end{cases} \tag{3.6}$$

Here and in what follows if $x \in \mathbb{Z}_p$ the symbol x^{-1} denotes its inverse in $\mathbb{Z}_p$, namely $x \cdot x^{-1} \equiv 1 \pmod{p}$. The inverse is unique because $\mathbb{Z}_p$ is a field since p is prime.

If $p = 4m+1$, -1 is a quadratic residue of p as we have already recalled; then we can denote by λ_p (or simply λ where the context is clear) the largest integer (mod p) such that

$$\lambda_p^2 \equiv -1 \pmod{p} \tag{3.7}$$

and we denote by Γ a representative of the equivalence relation in $\mathbb{Z}_p^*$:

$$x \sim y \Longleftrightarrow y = \lambda^s x \qquad \text{for } s \in \{1, 2, 3, 4\} . \tag{3.8}$$

We can write

$$\mathbb{Z}_N^* = \Gamma \cup (\lambda\Gamma) \cup (-\Gamma) \cup (-\lambda\Gamma) \tag{3.9}$$

and we can choose Γ in such a way that

$$\Gamma \cup (\lambda\Gamma) = \{1, \ldots, 2m\} . \tag{3.10}$$

Then we have:

Proposition 3.1. A complete system of orthogonal eigenvectors of the operator $\mathcal{F}_p$, where p is prime such that $p = 4m+1$, is given by

$$\begin{array}{ll} \psi_{\{\bar{k},0\}} & \text{with eigenvalue } 1 \\ \{\Phi_{j,r} : j \in \Gamma, r = 0, 1, 2, 3\} & \text{with eigenvalue } i^r \end{array} \tag{3.11}$$

where

$$\Phi_{j,r} = \frac{1}{2} \sum_{s=0}^{3} i^{-sr} \exp\left[\frac{\pi i}{p} \lambda j^2 \frac{1-\lambda^{rs}}{2}\right] \psi_{\bar{k},\lambda^r j} \qquad j \in \Gamma \quad r = 0, 1, 2, 3 \tag{3.12}$$

and $\bar{k} = \lambda/2$.

Proof. We have

$$\begin{aligned} (\mathcal{F}\psi_{\bar{k},j})_m &= \frac{1}{\sqrt{p}} \sum_{q=0}^{p-1} \exp \frac{2\pi i}{p} mq \exp \frac{2\pi i}{N} (\bar{k}q^2 + jp) \\ &= \epsilon_p \left(\frac{\bar{k}}{p}\right) \exp\left[-\frac{2\pi i}{p}(m+j)^2 + (4\bar{k})^{-1}\right] \\ &= \exp\left[\frac{2\pi i}{p}(\bar{k}j^2 + \bar{k}m^2 + 2m\bar{k}j)\right] \\ &= \exp\left[\frac{\pi i}{p}\lambda j^2\right] (\psi_{\bar{k},\lambda j})_m \end{aligned}$$

and, in general,

$$\mathcal{F}\psi_{\bar{k},\lambda^s j} = \exp\left[\frac{\pi i}{p}\lambda j^2 \lambda^{2s}\right]\psi_{\bar{k},\lambda^{s+1} j} \tag{3.13}$$

where we have used the relation

$$\left(\frac{\bar{k}}{p}\right) = \left(\frac{\lambda}{p}\right)\left(\frac{2^{-1}}{p}\right) = \left(\frac{\lambda}{p}\right)\left(\frac{2}{p}\right) = (-1)^{(N-1)/4}(-1)^{(p^2-1)/8}$$
$$= \begin{cases} 1\cdot 1 & p \equiv 1 \pmod 1 \\ (-1)(-1) & p \equiv 3 \pmod 4 \end{cases}$$

(see [8, theorems 9.4, 9.5]).

Then

$$\begin{aligned}\mathcal{F}\Phi_{j,r} &= \frac{1}{2}\sum_{s=0}^{3} i^{-sr}\exp\left[\frac{\pi i}{p}\lambda j^2\frac{1-\lambda^{2s}}{2}\right]\mathcal{F}\psi_{\bar{k},\lambda^s j} \\ &= \frac{1}{2}\sum_{s=0}^{3} i^{-sr}\exp\left[\frac{\pi i}{p}\lambda j^2\frac{1-\lambda^{2s}}{2}\right]\exp\left[\frac{\pi i}{p}\lambda j^2\lambda^{2s}\right]\psi_{\bar{k},\lambda^s j} \\ &= i^r\frac{1}{2}\sum_{s=0}^{3} i^{-(s+1)r}\exp\left[\frac{\pi i}{p}\lambda j^2\frac{1-\lambda^{2(s+1)}}{2}\right]\psi_{\bar{k},\lambda^{s+1} j} \\ &= i^r\Phi_{j,r}\,.\end{aligned}$$

The orthonormality of the eigenvectors is implied by the orthonormality of the vectors $\psi_{\bar{k},j}$ and a simple computation based on (3.5).

It is now straightforward to obtain a complete system of eigenvectors of the sine Fourier transform operator $S = C^{p-1} \longrightarrow C^{p-1}$ whose matrix elements are

$$(S)_{xy} = \frac{1}{\sqrt{p}}\sin\left(\frac{2\pi}{p}xy\right) \qquad x, y = 1, \ldots, p-1 \tag{3.14}$$

obtained by the discrete Fourier transform operator $\mathcal{F}_p$

$$S = \frac{\mathcal{F}_p - \mathcal{F}_p^*}{2i}\,. \tag{3.15}$$

Remark that *a priori* S is defined on $\mathbb{C}^p$. For the sake of simplicity we have eliminated the first row and the first column which are equal to zero and we thus consider it as an operator on $\mathbb{C}^{p-1}$.

We construct the eigenvalues of the operator S by means of linear combinations of the real vectors

$$\begin{aligned}&\frac{1}{\sqrt{p}}\cos\left(\frac{2\pi}{p}ax^2\right)\sin\left(\frac{2\pi}{p}bx\right) \\ &\frac{1}{\sqrt{p}}\sin\left(\frac{2\pi}{p}ax^2\right)\sin\left(\frac{2\pi}{p}cx\right)\end{aligned} \tag{3.16}$$

with a suitable choice of a, b, c. With the previous set of Γ and λ we have (the proof is a straightforward verification based on the action of $\mathcal{F}$ specified in proposition 3.1 and on (3.2)):

Proposition 3.2. Let p be a prime such that $p = 4m + 1$; then the spectrum of the the operator S consists of the eigenvalues $-1, 0, 1$, and a set of corresponding eigenvectors is specified as follows:

$$\begin{array}{lll} e_j(x) + e_{-j}(x) & j = 1, \ldots, 2m & \text{eigenvalue } 0 \\ \Phi^+_{k,p}(x) & k \in \Gamma & \text{eigenvalue } +1 \\ \Phi^-_{h,p}(x) & h \in \lambda\Gamma & \text{eigenvalue } -1\,. \end{array} \tag{3.17}$$

Here $e_j : j = 1, \ldots, p-1$ is the canonical basis of $\mathbb{R}^{p-1}$, $x = 1, \ldots, p-1$ and

$$\Phi^+_{k,p}(x) = \frac{1}{\sqrt{p}}\left[\cos\left(\frac{2\pi}{p}\lambda(2)^{-1}x^2\right)\sin\left(\frac{2\pi}{p}kx\right)\right] + \frac{1}{\sqrt{p}}\left[\sin\left(\frac{2\pi}{p}\lambda(2)^{-1}(k^2+x^2)\right)\sin\left(\frac{2\pi}{p}\lambda kx\right)\right]$$

$$\Phi^-_{h,p}(x) = \frac{1}{\sqrt{p}}\left[\cos\left(\frac{2\pi}{p}\lambda(2)^{-1}x^2\right)\sin\left(\frac{2\pi}{p}hx\right)\right] + \frac{1}{\sqrt{p}}\left[\sin\left(\frac{2\pi}{p}\lambda(2)^{-1}(h^2+x^2)\right)\sin\left(\frac{2\pi}{p}\lambda hx\right)\right].$$

Remark. (i) By standard estimates on Gauss sums (we omit the details) it can easily be seen that the above eigenvectors are normalized as follows:

$$\|\Phi^+_{k,p}\| = \|\Phi^-_{h,p}\| = 1 + \mathrm{O}\left(\frac{1}{\sqrt{p}}\right). \tag{3.18}$$

(ii) By the same argument of corollary 2.1, if $p = 2N + 1$ $(N = 2m)$ an eigenvector basis for J is given by

$$\begin{array}{llll} \Phi^+_{k,2N+1}(x) & k \in \Gamma & x = 1, \ldots, N & \text{eigenvalue } +1 \\ \Phi^-_{h,2N+1}(x) & h \in \lambda\Gamma & x = 1, \ldots, N & \text{eigenvalue } -1\,. \end{array} \tag{3.19}$$

(iii) The choice of the index $h \in \lambda\Gamma$ labelling the vectors Φ^-_h is due to following property of the eigenvector components:

$$\Phi^-_h(x) = \Phi^+_k(\lambda x) \qquad \text{if } h = \lambda k\,. \tag{3.20}$$

Different choices of index h (always in a Γ-type subset of $\mathbb{Z}_N$) generate analogous relations among the eigenvector components.

Let us now apply this construction to prove for that $p = 2N + 1$ prime of the form $4m + 1$ the matrix J does not admit any spin configuration among its eigenvectors. To see this, first remark that, by the same argument of lemma 2.1 and corollary 2.1, the vector $\chi_L\left(\frac{x}{p}\right) : x = 1, \ldots, N$ is in the kernel of S. Indeed we have

$$(S\chi_L)_k = \frac{1}{\sqrt{p}}\sum_{x=1}^{N}\sin\left(\frac{2\pi}{p}kx\right)\chi_L(x) = \frac{1}{2i\sqrt{p}}\left(\chi_L(k) - \chi_L(-k)\right)\sum_{x=1}^{p}\chi_L(x)\mathrm{e}^{(2\pi i/p)kx}$$

$$= \frac{1}{2i\sqrt{p}}\left(\chi_L(k) - \chi_L(-k)\right) i\sqrt{p} = 0$$

since $\chi_L(-1) = 1$ if $N = 4m + 1$ (see, e.g., [7, theorem 9.10]).

The second step is represented by the observation that, when p is a prime of the form $4m+1$, if a spin configuration is an eigenvector it cannot distinguish between the eigenvalue 1 or -1 of S (and hence of J), whose eigenspaces V^+ and V^- have one and the same dimension as we have seen above. This fact is the key difference with p prime of the form $4m + 3$: here the dimension of V^+ and V^- differs by one and the distinction is possible.

Lemma 3.1. Let $p = 2N + 1, N = 2m$, and once more denote by $V^\pm$ the subspaces corresponding to the eigenvalues ± 1 of S. Then there exists $v = \in V^+$, $v = (v_1, \dots, v_p)$, $v_k \in \{\pm 1\}$, $k = 1, \dots, p$ if and only there exists $u = \in V^-$, $u = (u_1, \dots, u_p)$, $u_k \in \{\pm 1\} k = 1, \dots, p$.

Proof. The vectors $\Phi^+_{k,p}(x)$ and $\Phi^-_{h,p}(x)$ defined in (3.19) span V^+ and V^-, respectively, and taken together with the basis of V^0 form a basis of C^p. Therefore if $v \in V^+$ there are coefficients c_k such that

$$v = \sum_{k \in \Gamma} c_k \Phi^+_{k,p} .$$

Now set

$$u = \sum_{h \in \lambda\Gamma} d_h \Phi^-_{h,p}$$

with $d_{\lambda k} = c_k$. The vector u is obviously the eigenvector of S corresponding to the eigenvalue -1. Moreover, since $\Phi^-_{h,p}(x) = \Phi^+_{k,p}(\lambda x)$, we have

$$u_x = \sum_{h \in \lambda\Gamma} d_h \Phi^-_{h,p}(x) = \sum_{k \in \Gamma} c_k \Phi^+_{k,p} = v_{\lambda x} .$$

Therefore $u_x \in \{\pm 1\} \Longleftrightarrow v_{\lambda x} \in \{\pm 1\} \Longleftrightarrow v_x \in \{\pm 1\}$ $x = 1, \dots, p - 1$ and this proves the lemma. □

Hence we have:

Proposition 3.3. If $p = 2N + 1, N = 2m$ no antisymmetric eigenvector (with eigenvalue ± 1) of the matrix S, and hence no eigenvector of J, can have all components ± 1.

Proof. Consider the numbers $\sin((2\pi/p)kx)$, $k, x = 1 \dots, p - 1$. Only $p - 1 = 4m$ of them are distinct. We can label them as $\mu_s = \sin((2\pi/p)s)$, $s = 1 \dots, \frac{1}{2}(p-1) = 4m$. These numbers are all irrational (see, e.g., [6, theorem 6.15]). Now the eigenvector relation $S\chi_L = 0$ yields, since the eigenvalue 0 has multiplicity $2m+1$, $p-2m-1 = \frac{1}{2}(p-1) = 2m$ independent relations with integer coefficients among the $4m$ numbers μ_s. By the antisymmetry of S, these conditions are necessarily equivalent to the standard reflection conditions

$$\mu_s = \sin\left(\frac{2\pi}{p} s\right) = -\sin\left(\frac{2\pi}{p}(p - s)\right) = \mu_{p-s} \qquad s = 1, \dots, 2m . \tag{3.21}$$

If there is an eigenvector $v = (v_1, \dots, v_p)$ with $v_s \in \{\pm 1\}$, $s = 1, \dots, p-1$, the eigenvector condition $Sv = v$ yields $p - 1 - m = 3m$ independent conditions with integer coefficients because the eigenvalue 1 has multiplicity m. Again, $2m$ of these conditions are simply (equivalent to) the conditions (3.21). We are thus left with m independent relations with integer coefficients among the $2m$ numbers μ_s. However, by lemma 3.1, the existence of v as above is equivalent to the existence of $u = (v_1, \ldots, u_p)$ with $u_s \in \{\pm 1\}$, $s = 1, \ldots, p-1$ such that $Sv = -v$. Therefore, since also the multiplicity of the eigenvalue -1 is m, we get other m independent relations with integer coefficients between the numbers μ_s. The m relations $Sv = v$ are independent of the m relations $Su = -u$ because otherwise the vectors u and v would have non-zero components along each other, thus contradicting the orthogonality between V^+ and V^-. We thus end up with $2m$ linearly independent relations with integer coefficients among the $2m$ numbers μ_s, and this contradicts their irrationality. This proves the statement as far as the matrix S is concerned, and the assertion for J follows immediately by antisymmetry. This proves the proposition. □

Remark. The above argument applies also to any $p = 4m + 1$ non-prime provided

(a) V^+ and V^- have the same dimension, and
(b) lemma 3.2 also holds for p non-prime.

Property (a) holds for S, and hence for J, because the eigenspaces of $\mathcal{F}$ corresponding to the eigenvalues i and $-i$ have one and the same dimension; however, we are unable to prove property (b), even though it looks natural, because the eigenvector construction of proposition 3.1 requires p prime.

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